UDC 539.3

ON THE EFFECT OF A NORMAL LOAD MOVING AT A CONSTANT VELOCITY ALONG THE BOUNDARY OF AN ELASTIC HALF-SPACE

PMM Vol. 41, № 1, 1977, pp. 134-142

V. A. CHURILOV
(Moscow)

(Received March 16, 1976)

The effect on an elastic isotropic half-space is studied of a load applied normal to the boundary and moving at a constant velocity along one of the coordinate axes. Underlying the search for the solutions of the elasticity theory equations in displacements in a coordinate system coupled to the moving load is the method of complex solutions [1]. However, the anisotropy of the elastic properties of the medium which hence occurs is inadequate to a direct utilization of this method, hence additional elastic parameters are introduced. The solutions hence found are then converted to an isotropic medium by a passage to the limit of the elastic parameters to the isotropic parameters.

Formulas are obtained for the displacements of the elastic half-space during motion of a normal distributed load bounded by an ellipse, by a concentrated force, by a system of forces bounded by any closed contour along its boundary.

1. Solutions of a system of second order linear differential equations of elliptic type. In the absence of mass forces, let us consider some hypothetical anisotropic medium which is described by the following system of equations in the rectangular coordinates x, y, z:

$$(1 + k_2{}^2S)\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} + p\frac{\partial^2 u_2}{\partial x\,\partial y} + p\frac{\partial^2 u_3}{\partial x\,\partial z} = 0 \qquad (1.1)$$

$$p\frac{\partial^2 u_1}{\partial x\,\partial y} + \frac{\partial^2 u_2}{\partial x^2} + (1 + S)\frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2} + S\frac{\partial^2 u_3}{\partial y\,\partial z} = 0$$

$$p\frac{\partial^2 u_1}{\partial x\,\partial z} + S\frac{\partial^2 u_2}{\partial y\,\partial z} + \frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} + (1 + S)\frac{\partial^2 u_3}{\partial z^2} = 0$$

Here u_j $(j = 1, 2, 3)$ are displacement components of the material points of the medium, and k_2, S, p are positive parameters characterizing the elastic properties of the medium and independent of the current coordinates.

Following the method elucidated in [1], and omitting the computations, it is possible to obtain the solution

$$u_j = \int_0^{2\pi} \operatorname{Re}\Big[\sum_{k=1}^{3} \Delta_k^{(j)} \omega_k(\Omega_k)\Big] d\theta, \quad j = 1, 2, 3 \qquad (1.2)$$

Here

$$\Delta_k^{(1)} = A_k^{(1)} l^3 n_k, \quad \Delta_k^{(2)} = A_k^{(2)} l^2 m n_k \qquad (1.3)$$

$$\Delta_1^{(3)} = A_1^{(2)} l^2 m^2 n_1{}^2, \quad \Delta_{2,3}^{(3)} = A_{2,3}^{(2)} l^2 n_{2,3}^2$$

$$A_k^{(1)} = p(S_k - 1), \quad A_k^{(2)} = p^2 - k_2{}^2 S^2 + S(S_k - 1) \qquad (1.4)$$

$$n_k = i\sqrt{S_k l^2 + m^2}, \quad k = 1, 2, 3 \qquad (1.5)$$

$$l = \cos\theta, \quad m = \sin\theta$$

$$S_1 = 1, \quad S_{2,3} = \frac{-M \mp \sqrt{M^2 - 4(1+S)(1+k_2^2 S)}}{2(1+S)}$$

$$M = 2 + S + k_2^2 S + k_2^2 S^2 - p^2$$

The functions ω_k, dependent on the argument

$$\Omega_k = xl + ym + n_k z \tag{1.6}$$

are arbitrary.

2. Finding the solutions for a half-space. Let us select the following

$$\frac{\sigma_x}{\mu} = k_2(1+S)\frac{\partial u_1}{\partial x} + \frac{p-k_2}{k_2}\left(\frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) \tag{2.1}$$

$$\frac{\sigma_y}{\mu} = (p-k_2)\frac{\partial u_1}{\partial x} + (1+S)\frac{\partial u_2}{\partial y} + (S-1)\frac{\partial u_3}{\partial z}$$

$$\frac{\sigma_z}{\mu} = (p-k_2)\frac{\partial u_1}{\partial x} + (S-1)\frac{\partial u_2}{\partial y} + (1-S)\frac{\partial u_3}{\partial z}$$

$$\frac{\tau_{xy}}{\mu} = k_2\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}, \quad \frac{\tau_{yz}}{\mu} = \frac{\partial u_3}{\partial y} + \frac{\partial u_2}{\partial z}$$

$$\frac{\tau_{xz}}{\mu} = k_2\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z}$$

as the relation between the stresses and strains.

If an axisymmetric normal load acts on the boundary of the half-space $z \geqslant 0$, and there are no tangential stresses, then by proceeding from (2.1) the following boundary conditions can be written:

$$(p-k_2)\frac{\partial u_1}{\partial x} + (S-1)\frac{\partial u_2}{\partial y} + (1+S)\frac{\partial u_3}{\partial z} = \frac{\sigma_z(r)}{\mu} \tag{2.2}$$

$$\frac{\partial u_3}{\partial y} + \frac{\partial u_2}{\partial z} = 0, \quad k_2\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z} = 0, \quad r = \sqrt{x^2 + y^2}$$

Differentiating (1.2) with respect to x, y, z, and substituting them into (2.2), we obtain the system of equations

$$\int_0^{2\pi} \mathrm{Re} \sum_{k=1}^{3} [(p-k_2)\Delta_k^{(1)} l + (S-1)\Delta_k^{(2)} m + (1+S)\Delta_k^{(3)} n_k] \times$$
$$\frac{d\omega_k}{d\xi}\, d\theta = \frac{\sigma_z(r)}{\mu}$$

$$\int_0^{2\pi} \mathrm{Re} \sum_{k=1}^{3} [\Delta_k^{(3)} m + \Delta_k^{(2)} n_k] \frac{d\omega_k}{d\xi}\, d\theta = 0$$

$$\int_0^{2\pi} \mathrm{Re} \sum_{k=1}^{3} [k_2\Delta_k^{(3)} l + \Delta_k^{(1)} n_k] \frac{d\omega_k}{d\xi}\, d\theta = 0$$

$$\xi = xl + ym$$

The method for finding the functions $d\omega_k / d\xi$ is elucidated in [2, 3]. Omitting the tedius computations, we write down the final result

$$\frac{d\omega_k}{d\xi} = \frac{\Delta_k}{\Delta_0}\frac{d\Psi^+(\xi)}{d\xi}$$

Here

$$\Delta_0 = A_1^{(2)}A_2^{(1)}A_3^{(2)}l^7\ mn_1n_2n_3\Delta \tag{2.3}$$

$$\Delta = \frac{1}{\gamma}(1-f)[1+(4\gamma-1)a-(a-1)f] \tag{2.4}$$

$$\gamma = \frac{S}{1+S}, \quad a = k_2^2l^2+m^2, \quad f = \sqrt{1+\gamma(a-1)}$$

$$\Delta_1 = 2[A_2^{(2)}A_3^{(1)} - A_2^{(1)}A_3^{(2)}]\,l^5mn_2^2n_3^2 \tag{2.5}$$

$$\Delta_2 = A_1^{(2)}[A_3^{(2)}k_2l^2 + A_3^{(1)}(m^2+1)]\,l^5mn_3^2$$

$$\Delta_3 = -A_1^{(2)}[A_2^{(2)}k_2l^2 + A_2^{(1)}(m^2+1)]\,l^5mn_2^2$$

$$\frac{d\Psi^+(\xi)}{d\xi} = \frac{1}{2\pi\mu}\frac{d}{d\xi}\int_0^\xi\frac{r\sigma_z(r)\,dr}{\sqrt{\xi^2-r^2}} \tag{2.6}$$

Continuing $d\Psi^+/d\xi$ analytically into the domain $z>0$ by the Cauchy formula

$$\frac{d\Psi}{d\Omega_k} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{d\Psi^+(\xi)}{d\xi}\frac{d\xi}{\xi-\Omega_k}$$

and integrating with respect to Ω_k, we finally obtain

$$\omega_k(\Omega_k) = \frac{\Delta_k}{\Delta_0}\Psi(\Omega_k) \tag{2.7}$$

Substituting (2.7) into (1.2) yields the displacement components

$$u_j = \int_0^{2\pi}\mathrm{Re}\left[\sum_{k=1}^{3}\frac{\Delta_k^{(j)}\Delta_k}{\Delta_0}\Psi(\Omega_k)\right]d\theta \tag{2.8}$$

Here $\Delta_k^{(j)}$, Δ_0, Δ_k are determined by means of (1.3), (2.3), (2.5), respectively.

3. Solutions for a half-space when the elastic parameter p equals the parameter k_2S. Let us obtain the solution of the system (1.1) with the boundary conditions (2.2) when $p = k_2S$.

The ratios $A_2^{(2)}/A_2^{(1)}$ and $A_3^{(1)}/A_3^{(2)}$, in which the parameter p enters, can be extracted in an examination of the expressions $\Delta_k^{(j)}\Delta_k/\Delta_0$ $(j, k = 1, 2, 3)$, which enter into (2.8). If (1.4) is substituted in these ratios and the indeterminacy is exposed for the first of them when p tends to k_2S, we obtain

$$A_2^{(2)}/A_2^{(1)} = A_3^{(1)}/A_3^{(2)} = k_2$$

Therefore, the quantities $\Delta_k^{(j)}\Delta_k/\Delta_0$ take on the following values:

$$\frac{\Delta_1^{(1)}\Delta_1}{\Delta_0} = 0, \quad \frac{\Delta_2^{(1)}\Delta_2}{\Delta_0} = \frac{2k_2ln_3}{i\Delta} \tag{3.1}$$

$$\frac{\Delta_3^{(1)}\Delta_3}{\Delta_0} = -\frac{k_2l(a+1)}{\Delta}, \quad \frac{\Delta_1^{(2)}\Delta_1}{\Delta_0} = -\frac{2(k_2^2-1)mn_3}{i\Delta}$$

$$\frac{\Delta_2^{(2)}\Delta_2}{\Delta_0} = \frac{2k_2^2mn_3}{i\Delta}, \quad \frac{\Delta_3^{(2)}\Delta_3}{\Delta_0} = -\frac{m(a+1)}{\Delta}$$

$$\frac{\Delta_1^{(3)}\Delta_1}{\Delta_0} = -\frac{2(k_2^2-1)m^2n_3}{\Delta}, \quad \frac{\Delta_2^{(3)}\Delta_2}{\Delta_0} = \frac{2k_2^2n_3}{\Delta}$$

$$\frac{\Delta_3^{(3)}\Delta_3}{\Delta_0} = -\frac{n_3(a+1)}{\Delta}$$

Formulas (1.5) have the form

$$n_1 = n_2 = i, \quad n_3 = if \tag{3.2}$$
$$S_1 = S_2 = 1, \quad S_3 = \frac{1+k^2{}_2S}{1+S}$$

Substituting (3.1),(3.2) into (2.8), we finally obtain

$$u_1 = \int_0^{2\pi} k_2 l \Psi_1(\theta)\,d\theta, \quad u_2 = \int_0^{2\pi} m\Psi_1(\theta)\,d\theta \tag{3.3}$$

$$u_3 = \int_0^{2\pi} \frac{f}{\Delta}\left[2a \operatorname{Re} i\Psi(\Omega_2) - (a+1)\operatorname{Re} i\Psi(\Omega_3)\right]d\theta$$

$$\Psi_1(\theta) = \frac{1}{\Delta}\left[2f \operatorname{Re}\Psi(\Omega_2) - (a+1)\operatorname{Re}\Psi(\Omega_3)\right]$$

4. Effect of a normal load distributed over the area of an ellipse on a half-space.

Example 1. Let D be the domain of the plane $z = 0$ bounded by an ellipse with semi-axes a_1, b. We consider the normal stresses at the points $M(x, y, 0)$ of the half-space boundary in the form

$$\sigma_z = \frac{k_2P_z}{2\pi a_1 b}\left(1 - \frac{x^2}{a_1^2} - \frac{y^2}{b^2}\right)^{-1/2}, \quad M \in D \tag{4.1}$$
$$\sigma_z = 0, \quad M \overline{\in} D$$

In order to use (2.6), a change of variables must be made and the first formula in (4.1) must be represented in the form

$$\sigma_z = \frac{k_2P_z}{2\pi R_1\sqrt{R_1^2 - r_1^2}}, \quad R_1 = \sqrt{a_1 b}, \quad r_1^2 = x_1^2 + y_1^2 \tag{4.2}$$
$$x = \sqrt{\frac{a_1}{b}}\,x_1, \quad y = \sqrt{\frac{b}{a_1}}\,y_1$$

Substituting (4.2) into (2.6), we find

$$\frac{d\Psi^+(\xi)}{d\xi_1} = -\frac{k_2P_z}{8\pi^2\mu R_1}\frac{d}{d\xi}\ln\frac{|\xi_1 - R_1|}{|\xi_1 + R_1|}$$

Continuing $d\Psi^+(\xi_1)/d\xi_1$ analytically in the domain $z > 0$ and integrating with respect to Ω_k, we find

$$\Psi(\Omega_k) = -\frac{k_2P_z}{8\pi^2\mu\sqrt{a_1b}}\ln\frac{\Omega_k - \sqrt{a_1b}}{\Omega_k + \sqrt{a_1b}} \tag{4.3}$$

The principal values are understood for the logarithms. On the boundary $z = 0$ we correspondingly obtain

$$\Psi^+(\xi) = -\frac{k_2P_z}{8\pi^2\mu\sqrt{a_1b}}\left(\ln\frac{|\xi - \sqrt{a_1b}|}{|\xi + \sqrt{a_1b}|} + \delta\pi i\right) \tag{4.4}$$

$$\delta = \begin{cases} 1, & \xi \in D \\ 0, & \xi \overline{\in} D \end{cases}$$

To seek the displacements in an elastic half-space, there remains to substitute (4.3) into (3.3) in which l must be replaced by $l\Delta_4^{-1}$, m by $m\Delta_4^{-1}$, $d\theta$ by $\Delta_4^{-2}d\theta$, where

$$\Delta_4^{-1} = \left(\frac{a_1}{b} l^2 + \frac{b}{a_1} m^2\right)^{-1/2}$$

We give here only the final formula for the settlement of the half-space boundary in the domain of load action (4.1), calculated by using the substitution of (4.4) into the third formula of (3.3)

$$u_3(x, y) = \frac{k_2 P_z}{2\pi\mu\sqrt{a_1 b}} \int_0^{\pi/2} \frac{(1+f)f}{1+(4\gamma-1)a-(a-1)f} \frac{d\theta}{\Delta_4} \tag{4.5}$$

Example 2. Let us examine the effect of a uniformly distributed normal load over the area of a circle in the coordinates (x_1, y_1).

The boundary conditions are

$$\sigma_z = \frac{k_2 P_z}{\pi R_1^2}, \quad r_1 < R_1 \tag{4.6}$$

$$\sigma_z = 0, \quad r_1 > R_1$$

According to (2.6) we have

$$\frac{d\Psi^+(\xi_1)}{d\xi_1} = \frac{k_2 P_z \delta}{2\pi^2\mu R_1^2}$$

$$\delta = \begin{cases} 1, & |\xi_1| < R_1 \\ 1 - \dfrac{\xi_1}{\sqrt{\xi_1^2 - R_1^2}}, & |\xi_1| > R_1 \end{cases}$$

Continuing $d\Psi^+(\xi_1) / d\xi_1$ analytically in the domain $z > 0$, integrating with respact to Ω_k and returning to the old x, y variables by means of (4.2), we obtain

$$\Psi(\Omega_k) = -\frac{k_2 P_z}{2\pi^2\mu} \frac{1}{\Omega_k + \sqrt{\Omega_k^2 - a_1 b}} \tag{4.7}$$

Substituting (4.7) into (3.3), we can obtain the displacement of the half-space points. We present only the vertical displacement formula

$$u_3(x, y, z) = -\frac{k_2 P_z}{2\pi^2\mu} \int_0^{2\pi} \frac{f}{\Delta} \operatorname{Re} \sum_{k=2}^{3} \frac{(-1)^k a(1+a^{2-k})}{\Omega_k + \sqrt{\Omega_k - a_1 b}} \, i d\theta \tag{4.8}$$

5. Effect of a normal load moving at a constant velocity along the boundary of an isotropic half-space. If we make the substitution $x = k_2 x_2$ in (1.1), (2.1) and (3.3) and set

$$p = k_2 S, \quad S = 1 + \frac{\lambda}{\mu}, \quad \frac{1}{k_i^2} = 1 - \frac{c^2}{c_i^2} \quad (i = 1, 2)$$

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_2 = \sqrt{\frac{\mu}{\rho}}, \quad c < c_2$$

where λ, μ, ρ are, respectively, the Lamé elastic constants and the density of the medium, c_1, c_2 are the propagation velocities of the longitudinal and transverse waves, and c is the velocity of surface load motion, then (1.1) and (2.1) will be the elasticity theory equations for an isotropic medium in a body coordinate system of the moving load, while (3.3) will be their solutions for the half-space $z \geqslant 0$. On the basis of this consideration, we obtain some results for an isotropic half-space in moving coordinates from

Sect. 4.

Replacing x in (4.1) and (4.5) by k_2x_2 and a_1 by k_2a_2, we obtain

$$\sigma_z = \frac{P_z}{2\pi a_2 b}\left(1 - \frac{x_2^2}{a_2^2} - \frac{y^2}{b^2}\right)^{-1/2}, \quad M \in D \tag{5.1}$$

$$\sigma_z = 0, \qquad M \,\overline{\in}\, D$$

$$u_3(x_2, y) = \frac{k_2 P_z}{2\pi\mu\sqrt{a_2 b}} \int\limits_0^{\pi/2} \frac{(1+f)f}{1+(4\gamma-1)a-(a-1)f}\frac{d\theta}{\Delta_4} \tag{5.2}$$

$$\Delta_4 = \left(\frac{k_2^2 a_2}{b} l^2 + \frac{b}{a_2} m^2\right)^{1/2}$$

As is seen from (5.2), the integrand depends only on $a = k_2^2 l^2 + m^2$, hence, the integral turns out to be constant. Therefore, the moving load (5.1) yields a constant settlement in the domain of its action.

When the load is distributed uniformly over the domain D of the $z = 0$ plane bounded by an ellipse with the semi-axes a_2, b, the formulas (4.6) and (4.8) take the following form:

$$\sigma_z = \frac{P_z}{\pi a_2 b}, \quad M(x_2, y, 0) \in D \tag{5.3}$$

$$\sigma_z = 0, \quad M(x_2, y, 0) \,\overline{\in}\, D$$

$$u_3(x_2, y, z) = -\frac{k_2 P_z}{2\pi^2\mu} \int\limits_0^{2\pi} \frac{f}{\Delta} \sum_{k=2}^{3} \frac{(-1)^k a(1+a^{2-k})}{\Omega_k + \sqrt{\Omega_k^2 - k_2 a_2 b}}\, i d\theta$$

6. Effect of a normal concentrated force moving with constant velocity on the boundary of an elastic half-space. If $a_2 b$ tends to zero in the integrand of (5.3), then the real part is extracted and the irrationality in the denominator is gotten rid of simultaneously, then the following result can be obtained:

$$u_3(x_2, y, z) = -\frac{k_2 P_z}{4\pi^2\mu} \int\limits_0^{2\pi} \sum_{k=2}^{3} \frac{(-1)^k a(1+a^{2-k}) f^{k-1} z L(\theta)\, d\theta}{(k_2^2 x_2^2 + S_k z^2) + 2k_2 x_2 y l m + (y^2+z^2) m^2} \tag{6.1}$$

$$L(\theta) = \frac{(a+1)^2 + 4af}{(a-1)[a^3 + (5-16\gamma)a^2 - 5a - 1]}$$

For $z = 0$ this integral can be evaluated by using residues. Let us introduce a new variable t by means of the formulas

$$l = \frac{t^2+1}{2t}, \quad m = \frac{t^2-1}{2it}, \quad d\theta = \frac{dt}{it} \tag{6.2}$$

Then

$$u_3(x_2, y, z) = \frac{k_2 P_z}{2\pi\mu} i \sum [\operatorname{res} F_2(t) + \operatorname{res} F_3(t)] \tag{6.3}$$

$$F_k(t) = \frac{(-1)^k a(1+a^{2-k}) f^{k-1} L(\theta) 4izt}{[(k_2x_2 - iy)^2 + (S_k - 1)z^2]t^4 + 2(r_2^2 + S_k z^2)t^2 + (k_2x_2 + iy)^2 + (S_k - 1)z^2}$$

$$k = 2, 3 \tag{6.4}$$

$$a = \frac{(k_2^2-1)t^4 + 2(k_2^2+1)t^2 + (k_2^2-1)}{4t^2}, \quad r_2^2 = k_2^2 x_2^2 + y^2 + z^2$$

The singular points of the function (6.4) are

$$t = \pm\sqrt{\frac{-r_2^2 - z^2 \pm 2zr_2}{(k_2x_2 - iy)^2}} \tag{6.5}$$

$$t = \pm\sqrt{\frac{-r_2^2 - S_3z^2 \pm 2\sqrt{S_3}zr_1}{(k_2x_2 - iy)^2 + (S_3 - 1)z^2}} \tag{6.6}$$

$$r_1 = \sqrt{k_1^2x_2^2 + y^2 + z^2}$$

$$t = \pm\sqrt{-1 + 2\beta \pm 2\sqrt{\beta^2 - \beta}}, \quad \beta = \frac{a_p - 1}{k_2^2 - 1} \tag{6.7}$$

where a_p $(p = 1,2,3,4)$ are the roots of the equations

$$a - 1 = 0, \quad a^3 + (5-16\gamma)a^2 - 5a - 1 = 0$$

The residues of the function (6.4) on the $z = 0$ plane with respect to the singularities (6.7) are zero. It remains to find the residues of the functions (6.4) with respect to the simple poles (6.5) and (6.6), respectively, which are within the circle $|t| < 1$, where only the plus sign must be kept under the radicals in (6.5) and (6.6) since $|t_i| < 1$ $(i = 1,2,3,4)$ for $z \geqslant 0$. Using the known formula

$$\operatorname*{res}_{t=t_0} f(t) = \frac{p(t_0)}{g'(t_0)}, \quad g'(t_0) \neq 0, \quad f(t) = \frac{p(t)}{g(t)}$$

we obtain

$$\operatorname*{res}_{t=t_1} F_2(t) = \operatorname*{res}_{t=t_2} F_2(t) = -\frac{af}{16\gamma r_2}\frac{t}{\Delta} \tag{6.8}$$

$$\operatorname*{res}_{t=t_3} F_3(t) = \operatorname*{res}_{t=t_4} F_3(t) = \frac{(a+1)f^2}{32\gamma r_1\sqrt{S_3}}\frac{t}{\Delta} \tag{6.9}$$

where Δ is determined by means of (2.4).

All the poles on the $z = 0$ plane coincide, i.e.

$$t = \frac{ir_2}{k_2x_2 - iy} \tag{6.10}$$

Substituting (6.10) into (6.2), we have the following relationships

$$a = \frac{k_2^2r^2}{r_2^2}, \quad f^2 = \frac{S_3r_1^2}{r_2^2}, \quad a - 1 = \frac{(k_2^2 - 1)y^2}{r_2^2} \tag{6.11}$$

$$1 - f = -\frac{\gamma(k_2^2 - 1)y^2}{r_2(r_2 + r_1\sqrt{S_3})}, \quad 1 + f = \frac{r_2 + r_1\sqrt{S_3}}{r_2}$$

$$\frac{2af}{r_2} - \frac{(a+1)f^2}{r_1\sqrt{S_3}} = \frac{(k_2^2 - 1)y^2r_1\sqrt{S_3}}{r_2^4}, \quad S_3 = \frac{k_2^2}{k_1^2}$$

Substituting the doubled results (6.8) and (6.9) into the right side of (6.3) with (6.11) taken into account, and performing the necessary manipulations, we finally obtain the settlement of the boundary surface of the elastic half-space in the moving system of coordinates

$$u_3(x_2, y) = \frac{k_2P_z}{2\pi\mu}\frac{(1 + A)A}{4\gamma B + (1 - B)(1 + A)}\frac{1}{r_2} \tag{6.12}$$

$$\gamma = \frac{\lambda + \mu}{\lambda + 2\mu}, \quad A = \frac{r_1}{r_2}\frac{k_2}{k_1}, \quad B = \frac{k_2^2(x^2 + y^2)}{r_2^2}$$

$$r_1 = \sqrt{k_1^2x_2^2 + y^2}, \quad r_2 = \sqrt{k_2^2x_2^2 + y^2}$$

It should be noted that the estimate of the vertical displacement given in [4] is close to the exact displacement according to (6.12), when the velocity of the applied force is less than the velocity of the Rayleigh wave.

If a system of forces $P_z = q(\xi_2, \eta_1)\,d\Omega$ acts on the elastic half-space, then the settlement in the boundary $\Omega\ (x_2, y)$ is

$$u_3(x_2, y) = \iint_\Omega K(x_2 - \xi_2, y - \eta_1)\, q(\xi_2, \eta_1)\, d\xi_2 d\eta_1 \qquad (6.13)$$

Here

$$K(x_2 - \xi_2, y - \eta_1) = \frac{k_2 (1 + A) A}{2\pi\mu\, [4\gamma B + (1 - B)(1 + A)]} \frac{1}{r_2}$$

$$A = \frac{k_2}{k_1} \sqrt{\frac{k_1^2 (x_2 - \xi_2)^2 + (y - \eta_1)^2}{k_2^2 (x_2 - \xi_2)^2 + (y - \eta_1)^2}}, \qquad B = \frac{k_2^2 [(x_2 - \xi_2)^2 + (y - \eta_1)^2]}{k_2^2 (x_2 - \xi_2)^2 + (y - \eta_1)^2}$$

Let us consider an example. Let a load of the form

$$q(\xi_2, \eta_1) = \frac{P_z}{2\pi R^2}\left(1 - \frac{\xi_2^2 + \eta_1^2}{R^2}\right)^{-1/2}$$

act on the half-space boundary. We calculate the settlement due to this load at the point $x_2 = y = 0$. To do this, let us go over to a polar coordinate system

$$\xi_2 = \rho \cos\varphi, \qquad \eta_1 = \rho \sin\varphi$$

Then it follows from (6.13)

$$u_3(0,0,0) = \frac{P_z}{4\pi^2 \mu R^2} \int_0^R \left(1 - \frac{\rho^2}{R^2}\right)^{-1/2} d\rho \int_0^{2\pi} \frac{A B^{1/2} (1 + A)\, d\varphi}{4\gamma B + (1 - B)(1 + A)} \qquad (6.14)$$

$$A = \frac{k_2}{k_1} \frac{\sqrt{k_1^2 \cos^2\varphi + \sin^2\varphi}}{\sqrt{k_2^2 \cos^2\varphi + \sin^2\varphi}}, \qquad B = \frac{k_2^2}{k_2^2 \cos^2\varphi + \sin^2\varphi}$$

Let us go from the polar coordinate φ to another by means of the replacement

$$\operatorname{tg}\varphi = k_2 \operatorname{ctg}\theta$$

From this relationship we will have

$$\cos^2\varphi = \frac{m^2}{a}, \qquad \sin^2\varphi = \frac{k_2^2 l^2}{a}, \qquad d\varphi = -\frac{k_2 d\theta}{a}$$

Substituting these formulas and

$$\int_0^R \left(1 - \frac{\rho^2}{R^2}\right)^{-1/2} d\rho = \frac{\pi R}{2}$$

into (6.14), we obtain

$$u_3(0,0,0) = \frac{k_2 P_z}{2\pi\mu R} \int_0^{\pi/2} \frac{(1 + f) f}{1 + (4\gamma - 1) a - (a - 1) f} \frac{d\theta}{\sqrt{a}}$$

This result agrees with (5.2) if we set $a_2 = b = R$ therein.

REFERENCES

1. Sveklo, V. A., Elastic vibrations of an anisotropic body. Uchen. Zap. Leningrad Univ., № 17, 1949.
2. Sveklo, V. A., Boussinesq type problems for the anisotropic half-space. PMM Vol. 28, № 5, 1964.
3. Sveklo, V. A., The action of a stamp on an elastic anisotropic half-space. PMM Vol. 34, № 1, 1970.
4. Eason, G., The stresses produced in a semi-infinite solid by a moving surface force. Internat. J. Engng. Sci., Vol. 2, № 6, 1965.

Translated by M. D. F.